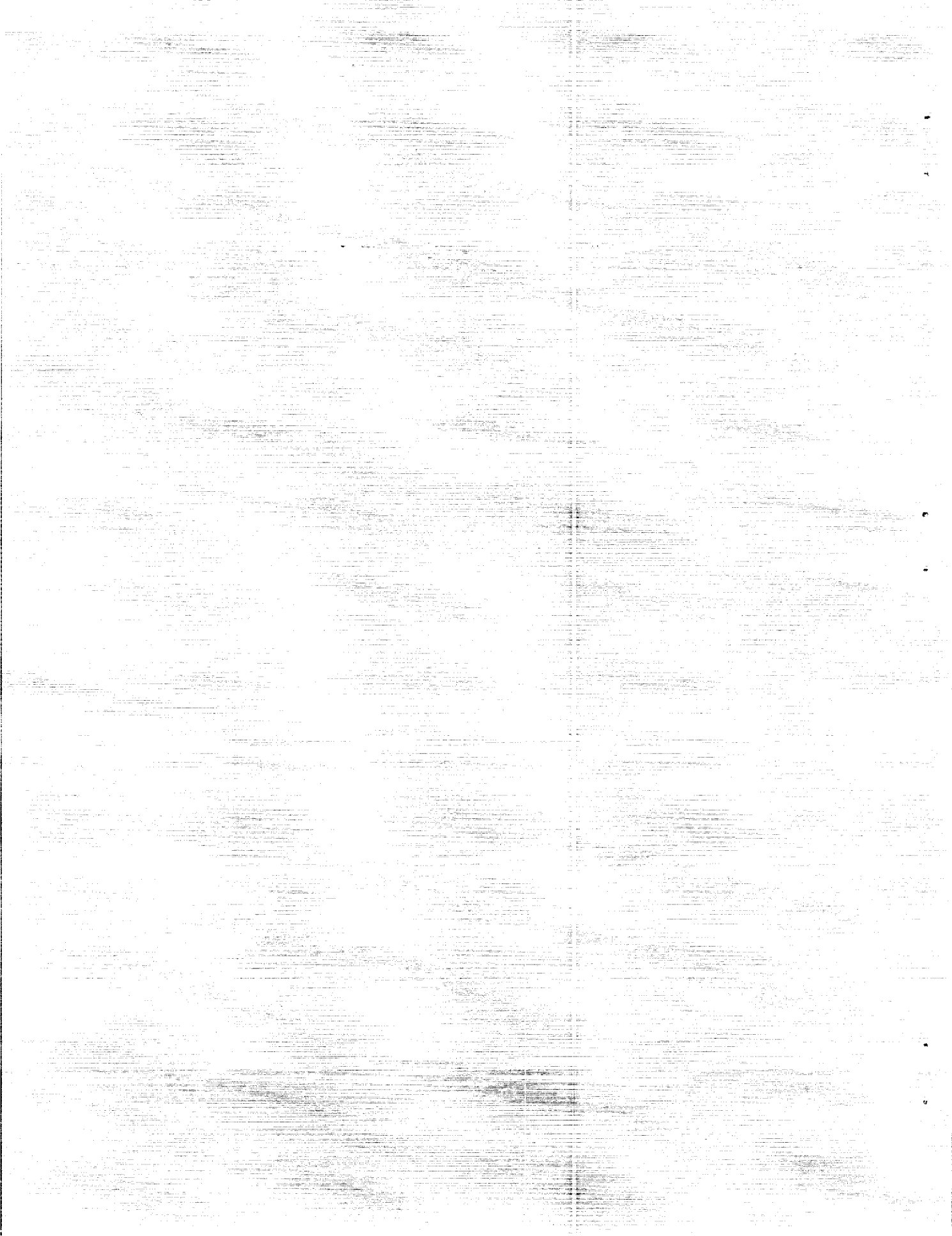*IN-34*
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TECHNICAL TRANSLATION

F-28

BOUNDARY-LAYER EQUATIONS AND THEIR BOUNDARY CONDITIONS

IN THE CASE OF MOTION AT SUPERSONIC VELOCITIES

IN A MODERATELY RAREFIED GAS

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BOUNDARY-LAYER EQUATIONS AND THEIR BOUNDARY CONDITIONS

IN THE CASE OF MOTION AT SUPERSONIC VELOCITIES

IN A MODERATELY RAREFIED GAS*

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In motion at supersonic velocities at great heights, the mean free path of the molecules 1 will be of the order of magnitude of the dimensions of the moving body L. We shall call a gas in which the path length 1 is short compared to L a moderately rarefied gas.

Following Prandtl's boundary-layer theory, equations are derived for a moderately rarefied gas which differ from the ordinary Prandtl equations by the presence of supplementary terms that contain higher derivatives of velocity and temperature; in the equations derived the normal pressure gradient differs from zero and is expressed by supplementary terms. The boundary conditions for these equations are found with the aid of kinetic theory; they are a generalization of Maxwell's /1/ and Smoluchowski's /2/ conditions for motion at a supersonic velocity. The velocity and height limits of applicability of the equations derived are set forth.

1. **Boundary Layer Equations in Moderately Rarefied Gases**. In our computations it will be convenient to employ the equations for rarefied gases in the form suggested by Chapman /3/; though we should note that Chapman made

a mistake[1] in calculating P_2, the tensor of viscous stresses in the second approximation.

We shall use the correct expression for P_2 and write the equations of motion for a rarefied gas in full, confining ourselves to the case of plane-parallel flow (the two-dimensional state) with no external forces present. We then get two equations:

$$\rho \frac{\partial u_x}{\partial t} + \rho u_x \frac{\partial u_x}{\partial x} + \rho u_y \frac{\partial u_x}{\partial y} = - \frac{\partial p}{\partial x} - \frac{\partial P_{xx}}{\partial x} - \frac{\partial P_{xy}}{\partial y} \qquad (1.1)$$

$$\rho \frac{\partial u_y}{\partial t} + \rho u_x \frac{\partial u_y}{\partial x} + \rho u_y \frac{\partial u_y}{\partial y} = - \frac{\partial p}{\partial y} - \frac{\partial P_{xy}}{\partial x} - \frac{\partial P_{yy}}{\partial y} \qquad (1.2)$$

In these equations P_{xx}, P_{xy}, and P_{yy} are given by the expressions

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[1] In the final expression for $p^{(2)}$ /tr. for Eq. (6) on p. 265/ the second term from the right should be written as follows:

$$k_2 \frac{\mu^2}{p} \left[\overline{\frac{\partial}{\partial x_i} \left(F_j - \frac{1}{\rho} \frac{\partial p}{\partial x_j} \right)} - \overline{\frac{\partial u_i}{\partial x} \frac{\partial u_j}{\partial k_k}} - 2 \overline{e_{ik} \frac{\partial u_k}{\partial x_j}} \right] \qquad (1)$$

where the tensor $\overline{\dfrac{\partial}{\partial x_i} \left(F_j - \dfrac{1}{\rho} \dfrac{\partial p}{\partial x_j} \right)}$ has the following value:

$$\overline{\frac{\partial}{\partial x_i} \left(F_j - \frac{1}{\rho} \frac{\partial p}{\partial x_j} \right)} = \frac{1}{2} \left[\frac{\partial}{\partial x_i} \left(F_j - \frac{1}{\rho} \frac{\partial p}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left(F_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \right) \right] -$$

$$- \frac{1}{3} \frac{\partial}{\partial x_k} \left(F_k - \frac{1}{\rho} \frac{\partial p}{\partial x_k} \right) \delta_{ij}$$

Tsien /4/, citing Chapman's final result for $p^{(2)}$, did not notice this mistake; moreover, Tsien's paper contains still another mistake. He writes the second term in Eq. (8) without a line above it.

In the translation of Tsien's paper, not only were these mistakes left uncorrected, but a new one was added: the line above the $\overline{\tau_{ij}}$ was left out in the term for k_3 in Eq. (11).

The authors who employed Chapman's equations /3/, usually taken from the reference /4/, naturally obtained false results, which were not immediately apparent because they were solving the general problems; they obtained, for example, the dimensionless form of these equations /5/, the integrals of one-dimensional motion /6/, etc.

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$$P_{xx} = -\frac{2}{3}\,\mu\left(2\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) + \frac{\mu^2}{p}\left[\frac{\vartheta_1}{3}\left(2\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y} - \frac{\partial u_y}{\partial y}\frac{\partial u_y}{\partial y}\right) - \right.$$

$$- \frac{\vartheta_2}{3}\left(6\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial p}{\partial x} - 3\frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{14}{3}\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial x} - 2\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y} + \right.$$

$$\left. + 2\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial y}\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial y} - \frac{7}{3}\frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial x}\right) +$$

$$+ \frac{\vartheta_6}{3}\left(\frac{7}{9}\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial x} - \frac{2}{9}\frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} + \frac{1}{4}\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial y} + \frac{1}{4}\frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial x} + \right.$$

$$\left.\left. + \frac{1}{2}\frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x}\right)\right] + \vartheta_3\frac{\mu^2}{3\rho T}\left(2\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2}\right) + \vartheta_4\frac{\mu^2}{3\rho p T}\left(2\frac{\partial p\,\partial T}{\partial x\,\partial x} - \right.$$

$$\left. - \frac{\partial p\,\partial T}{\partial y\,\partial y}\right) + \vartheta_5\frac{\mu^2}{3\rho T^2}\left[2\left(\frac{\partial T}{\partial x}\right)^2 - \left(\frac{\partial T}{\partial y}\right)^2\right]$$

$$(1.3)$$

$$P_{xy} = -\mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{\mu^2}{p}\left[\frac{\vartheta_1}{2}\left(\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial y} + \right.\right.$$

$$\left. + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y}\right) - \frac{\vartheta_2}{3}\left(\frac{3}{2}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{3}{2}\frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial p}{\partial x} + 4\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \right.$$

$$\left. + 4\frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} + 2\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial x} + 2\frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial y}\right) + \frac{\vartheta_6}{2}\left(\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \right.$$

$$\left.\left. + \frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial x} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y}\right)\right] + \vartheta_3\frac{\mu^2}{\rho T}\frac{\partial^2 T}{\partial x\,\partial y} +$$

$$+ \vartheta_4\frac{\mu^2}{2\rho p T}\left(\frac{\partial p}{\partial x}\frac{\partial T}{\partial y} + \frac{\partial p}{\partial y}\frac{\partial T}{\partial x}\right) + \vartheta_5\frac{\mu^2}{\rho T^2}\frac{\partial T}{\partial x}\frac{\partial T}{\partial y}$$

$$(1.4)$$

The expression for P_{yy} is readily obtained from Eq. (1.3) by substituting $\underline{y}$ for $\underline{x}$ and u_y for u_x. ϑ_i $(i = 1, 2, 3, 4, 5, 6)$ are the constants for definite models of molecular interaction. For the Maxwellian molecules:

$$\vartheta_1 = \frac{4}{3}\left(\frac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right), \qquad \vartheta_2 = 2, \qquad \vartheta_3 = 3 \qquad (1.5)$$

$$\vartheta_4 = 0, \qquad \vartheta_5 = \frac{3T}{\mu}\frac{d\mu}{dT}, \qquad \vartheta_6 = 8$$

For molecules that are solid elastic spheres:

$$\vartheta_1 = 1.014\frac{4}{3}\left(\frac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right), \qquad \vartheta_2 = 1.014 \cdot 2, \qquad \vartheta_3 = 0.806 \cdot 3$$

$$\vartheta_4 = 0.681, \qquad \vartheta_5 = 0.806\frac{3T}{\mu}\frac{d\mu}{dT} - 0.990, \qquad \vartheta_6 = 0.982 \cdot 8 \qquad (1.6)$$

The energy equation for a rarefied gas in two-dimensional flow is written as:

$$\rho\left(\frac{\partial}{\partial t} + u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)\left(\frac{u_x^2}{2} + \frac{u_y^2}{2} + C_p T\right) = \frac{\partial p}{\partial t} - \frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} -$$

$$- \frac{\partial}{\partial x}(P_{xx}u_x + P_{xy}u_y) - \frac{\partial}{\partial y}(P_{xy}u_x + P_{yy}u_y) \qquad (1.7)$$

where P_{xx}, P_{xy}, and P_{yy} are given by Eq. (1.3) and (1.4), while q_x is given by the expression

$$q_x = -\lambda \frac{\partial T}{\partial x} + \theta_2 \frac{\mu^2}{\rho T} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\frac{\partial T}{\partial x} +$$

$$+ \theta_2 \frac{\mu^2}{\rho T}\left\{\frac{2}{3}\frac{\partial}{\partial x}\left[T\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\right] + 2\left(\frac{\partial u_x}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial u_y}{\partial x}\frac{\partial T}{\partial y}\right)\right\} + \qquad (1.8)$$

$$+ \left(\vartheta_3 \frac{\mu^2}{\rho p}\frac{\partial p}{\partial x} + \theta_4 \frac{\mu^2}{\rho}\frac{\partial}{\partial x} + \theta_5 \frac{\mu^2}{\rho T}\frac{\partial T}{\partial x}\right)\left(\frac{2}{3}\frac{\partial u_x}{\partial x} - \frac{1}{3}\frac{\partial u_y}{\partial y}\right) +$$

$$+ \left(\theta_3 \frac{\mu^2}{\rho T}\frac{\partial p}{\partial y} + \theta_4 \frac{\mu^2}{\rho}\frac{\partial}{\partial y} + \theta_5 \frac{\mu^2}{\rho T}\frac{\partial T}{\partial y}\right)\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

The expression for q_y is obtained from Eq. (1.8) by simply substituting $\underline{y}$ for $\underline{x}$ and u_y for u_x. θ_i (i = 1, 2, 3, 4, 5) are constants given by the equations:

$$\theta_1 = \frac{15}{4}\left(\frac{7}{\mu} - \frac{T}{\mu}\frac{d\mu}{dT}\right), \qquad \theta_2 = \frac{45}{8}, \qquad \theta_3 = -3$$

$$\theta_4 = 3, \qquad \theta_5 = \frac{3}{2}\left(5 - \frac{T}{\mu}\frac{d\mu}{dT}\right) \qquad (1.9)$$

For Maxwellian molecules $\mu \sim T$, for molecules that are solid elastic spheres $\mu \sim \sqrt{T}$, and for air $\mu \sim T^{0.76}$.

The system of equations (1.1) - (1.4), (1.7), and (1.8), serves as a basis for securing the boundary layer equations of moderately rarefied gases. We shall employ Prandtl's method.

We shall assume that δ, the thickness of the boundary layer, is small compared to $\underline{L}$, the dimensions of the body, and that the viscosity and inertial forces in the boundary layer are of the same order of magnitude. Then evaluating the magnitude of all the terms in the equations of motion and of energy and neglecting terms whose relationship to the rest of the terms is of the order of $(\delta/L)^2 \ll 1$ we get:

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} - \frac{\partial P_{xx}}{\partial x} - \frac{\partial P_{xy}}{\partial y} \qquad (1.10)$$

$$0 = -\frac{\partial p}{\partial y} - \frac{\partial P_{yy}}{\partial y} \qquad (1.11)$$

$$\rho\left(\frac{\partial}{\partial t} + u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)\left(\frac{u_x^2}{2} + C_p T\right) = \frac{\partial p}{\partial t} - \frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} -$$

$$- \frac{\partial}{\partial x}(u_x P_{xx}) - \frac{\partial}{\partial y}(u_x P_{xy} + u_y P_{yy}) \qquad (1.12)$$

where

$$P_{xx} = \frac{\mu^2}{p}\left(\frac{\theta_5}{12} - \frac{2\theta_2}{3}\right)\left(\frac{\partial u_x}{\partial y}\right)^2 - \frac{\theta_5}{3}\frac{\mu^2}{\rho T^2}\left(\frac{\partial T}{\partial y}\right)^2 \qquad (1.13)$$

$$P_{xy} = -\mu\frac{\partial u_x}{\partial y} + \frac{\mu^2}{p}\left[\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y}\left(\frac{\vartheta_1}{2} + \frac{\vartheta_6}{2} - \frac{2\vartheta_2}{3}\right) + \right. \tag{1.14}$$

$$\left. + \frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial y}\left(\frac{\vartheta_1}{2} + \frac{\vartheta_6}{2} - \frac{5\vartheta_4}{3}\right) - \frac{\vartheta_2}{2}\frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial p}{\partial x}\right] + $$

$$+ \vartheta_3\frac{\mu^2}{\rho T}\frac{\partial^2 T}{\partial x\partial y} + v_4\frac{\mu^2}{2\rho\rho T}\frac{\partial p}{\partial x}\frac{\partial T}{\partial y} + \vartheta_5\frac{\mu^2}{\rho T^2}\frac{\partial T}{\partial x}\frac{\partial T}{\partial y}$$

$$P_{yy} = \frac{\mu^2}{p}\left(\frac{\vartheta_6}{12} - \frac{2\vartheta_2}{3}\right)\left(\frac{\partial u_x}{\partial y}\right)^2 + \frac{2\vartheta_3}{\rho T}\frac{\partial^2 T}{\partial y^2} + \frac{2\vartheta_5}{3\rho T^2}\left(\frac{\partial T}{\partial y}\right)^2 \tag{1.15}$$

$$q_x = \vartheta_6\frac{\mu^2}{2\rho}\frac{\partial^2 u_x}{\partial y^2} + \vartheta_5\frac{\mu^2}{2\rho T}\frac{\partial u_x}{\partial y}\frac{\partial T}{\partial y} \tag{1.16}$$

$$q_y = -\lambda\frac{\partial T}{\partial y} + \vartheta_1\frac{\mu^2}{\rho T}\frac{\partial T}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \vartheta_2\frac{2\mu^2}{3\rho T}\left[\frac{\partial T}{\partial y}\left(\frac{\partial u_x}{\partial x} + 4\frac{\partial u_y}{\partial y}\right) + \right.$$

$$\left. + T\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\right] + \vartheta_3\frac{\mu^2}{2\rho\rho}\frac{\partial p}{\partial x}\frac{\partial u_x}{\partial y} + \vartheta_4\frac{\mu^2}{6\rho}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + 4\frac{\partial u_y}{\partial y}\right) + $$

$$+ \vartheta_5\frac{\mu^2}{\rho T}\left[\frac{1}{2}\frac{\partial T}{\partial x}\frac{\partial u_x}{\partial x} + \frac{1}{3}\frac{\partial T}{\partial y}\left(2\frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial x}\right)\right] \tag{1.17}$$

This system (1.10) - (1.17) is incomplete; we shall supplement it with the equation of continuity:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) = 0 \tag{1.18}$$

and by the equation of state in the form of Clapeyron's law:

$$p = \rho RT \tag{1.19}$$

The system of equations (1.10) - (1.19) of the boundary layer of moderately rarefied gases differs from the ordinary Prandtl equations in having some additional terms; moreover, in this problem the normal pressure gradient is not equal to zero. The origin of this may be explained as follows: In the ordinary Prandtl equations the order of magnitude of the terms in the equation containing $\partial p/\partial y$ equals $\rho u_0^2\delta/L^2$, while in the equation containing $\partial p/\partial x$, it is equal to $\rho u_0^2/L$; their ratio is $\delta/L < 1$; but this means that the pressure gradient along the normal to the surface of the streamlined body is small compared to the longitudinal pressure gradient and may be written $\partial p/\partial y = 0$ with sufficient accuracy. In the case of rarefied gases the order of magnitude of $\partial p/\partial y$ is given by the Barnetta-Chapman additional terms, which, generally speaking, may be of an order of magnitude close to the order of magnitude of the principal terms in the equation for $\partial p/\partial x$ (a comparative evaluation of these terms is given below), so that the equation for $\partial p/\partial y$ may be written in the form of (1.11).

Hence, the rarefaction effect likewise results in the pressure within the boundary layer varying along the normal to the surface of the body, naturally differing from the pressure existing at the outer boundary of the boundary layer.

It should be noted that independently of us, Krzywoblocki /7/ recently employed an analogous method for simplifying the equations similar to the

Barnetta-Chapman equations. The equations employed by him as a basis were derived in the paper by Wang-Chang and Uhlenbeck /8/, with which we were unfortunately unfamiliar. Krzywoblocki retained the terms not only of unit order, but of the order of thickness of the boundary layer. The equations he derived are considerably more complicated than the system (1.10) - (1.17). Subsequently Krzywoblocki endeavored to solve his equations /9/, though the generalized relationships he derived still require further investigation.

2. Region of Applicability of the Equations Derived.

Let us compare the terms additionally considered in the equations of motion and energy with the principal terms as well as with the terms discarded in the Navier-Stokes' equation in effecting the transition to boundary layer equations. The ratio of the additional term to the principal term is of the order of magnitude:

$$\frac{\partial}{\partial y}\frac{\mu^2}{p}\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} : \frac{\partial}{\partial y}\mu\frac{\partial u_x}{\partial y} \sim \frac{\mu u_0}{pL} \sim \frac{pp\,\text{therm}\,lu_0}{pv^2\text{therm}}\frac{lu_0}{L} \sim M\frac{l}{L} \qquad (2.1)$$

where $M \equiv u_0/a$ (a is the velocity of sound, and $\underline{a} \sim v_{\text{therm}}$).

The order of magnitude of the ratio of the additional term to the discarded term will be:

$$\frac{\partial}{\partial y}\frac{\mu^2}{p}\left[\frac{\partial u_x}{\partial y}\right] : \frac{\partial}{\partial y}\mu\frac{\partial u_y}{\partial y} \sim \frac{\mu u_0 l}{p\delta^2} \sim \frac{lL}{\delta^2}M \sim M^2 \qquad (2.2)$$

Here it has been borne in mind that the viscosity and inertial forces will be of the same order of magnitude within the boundary layer and, hence:

$$\rho u_x\frac{\partial u_x}{\partial x} \sim \frac{\partial}{\partial y}\mu\frac{\partial u_x}{\partial y}, \qquad \frac{l^2}{\delta^2} \sim M\frac{l}{L} \qquad (2.3)$$

Thus the ratio of the additional terms to the principal terms will be of the order of magnitude of Ml/L, and will be M^2 with respect to the terms discarded in the theory of the boundary layer. Bearing in mind further that in conformity with (2.3):

$$Ml/L < 1 \qquad (2.4)$$

since $l/\delta < 1$ (otherwise there is no sense in speaking of a boundary layer), it may be stated that the boundary layer equations for moderately rarefied gases we have derived are applicable only for $M > 1$, i.e. for supersonic flows, and that the terms considered additionally are smaller than the basic Prandtl terms.

These evaluations of the additional terms enable us to delimit the region of applicability of the derived equations in terms of height and values of $\underline{M}$. The lower boundary for $\underline{M}$ is given by the inequality $M > 1$, and the upper boundary by $M < 10$, for high temperature develops in the boundary layer at $M \geq 10$, which results in dissociation of the gas molecules, not allowed for in the equations derived.

The evaluation in terms of height may be made as follows: the order of

magnitude of the ratio of the additional terms to the main Ml/L; according to (2.3) the magnitude of Ml/L must be at least one order of magnitude smaller than unity, but if this ratio is less than 1%, there is no sense in considering the additional terms. This consideration leads to the inequality:

$$0.01 < M\frac{l}{L} < 0.1 \qquad (2.5)$$

If we make use of the relationship between $\underline{l}$, the mean free path of the molecules, and $\underline{H}$, the height, taken from references (10) and (11), we get the following height ranges for given values of $\underline{M}$ and $\underline{L}$:

at $L = 100$ cm

$M =$	1	2	3	4	5	7	10
H км $=$	87 — 100,	78—94	70—92,	66—91,	65—90,	64—89,	63—83

at $L = 1000$ cm

$M =$	1	2	3	4	5	7	10
H км —	100—114,	94—112,	92—108,	91=105,	90—103,	89—101,	83—100

This shows that for large values of $\underline{M}$ the additional terms begin to make themselves felt at lower heights.

To be sure, the boundaries of the region of applicability obtained on the basis of the considerations cited above are highly uncertain, and the sole criterion of their correctness must be experiment.

3. <u>Boundary Conditions in a Moderately Rarefied Gas</u>. The ordinary boundary conditions for adhesion do not apply in a rarefied gas; numerous theoretical and experimental researches indicate that the gas slides over the surface of the body in this case. Let us find the boundary conditions, allowing for this slippage.

We shall make use of Maxwell's kinetic method, in which the complicated picture of the interactions between the molecules of the gas and the surface is replaced by a simplified model, based on two assumptions.

The first assumption is that the molecules of the gas, possessing random thermal velocity in addition to their forward velocity in the direction of flow, can collide with the surface of the body. Then a part α of all the incident molecules are adsorbed at the surface and as the result of some collisions with the molecules of the body acquire a velocity corresponding to the velocities in the quiescent gas at the temperature of the body, after which they are evaporated at these velocities in arbitrary directions with respect to the surface, while the portion $1 - \alpha$ is reflected from the surface, i.e., preserving the tangential component of velocity and changing the sign of the normal component.

The second assumption is that the distribution function of the molecular velocities close to the wall is taken to be the same as at a distance from it. The inadequacy of this assumption was understood by Maxwell himself, who pointed out that close to the wall some sort of discontinuity conditions

should exist for the distribution function.

These assumptions yield the following boundary conditions.

For the velocity:

$$(1-\alpha)\int_{-\infty}^{+\infty} dv_x \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{0} v_x v_y f \, dv_y + \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_z \int_{0}^{\infty} v_x v_y f \, dv_y =$$

$$= -u \left[(1-\alpha)\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{0} v_y f \, dv_y + \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_z \int_{0}^{\infty} v_y f \, dv_y \right] \quad (3.1)$$

where $\underline{u}$ is the velocity of the surface with respect to the gas in contact with it; $\underline{f}$ is the distribution function; and v_x, v_y, and v_z are the components of the random thermal velocities of the molecules.

For the temperature:

$$\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_z \int_{0}^{\infty} v_y f v^2 dv_y + (1-\alpha)\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{0} v_y f v^2 dv_y =$$

$$= \frac{6kT_w}{m}\left\{ \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_z \int_{0}^{\infty} v_y f \, dv_y + (1-\alpha)\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{0} v_y f \, dv_y \right\} \quad (3.2)$$

where T_w is the temperature of the surface; $\underline{k}$ is Boltzmann's constant; and $\underline{m}$ is the mass of the gas molecules.

The only unknown in Eq. (3.1) and (3.2) is the distribution function $\underline{f}$. Assigning it some approximate value, we obtain the explicit expression for velocity and temperature in terms of the hydrodynamic characteristics of flow.

Substituting the distribution function for the state of equilibrium yields the usual boundary conditions:

$$u = 0, \qquad T = T_w \qquad \text{at } y = 0 \qquad\qquad (3.3)$$

If we substitute a function characterizing a slight departure from the state of equilibrium in Eq. (3.1) and (3.2) as the distribution function, the boundary conditions for velocity and temperature may be written as follows, after a few transformations:

$$u_x = \frac{2-\alpha}{\alpha}\,\frac{\mu}{2}\left(\frac{2\pi}{p\rho}\right)^{1/2}\frac{\partial u_z}{\partial y} \quad . \qquad \text{at } y = 0 \qquad\qquad (3.4)$$

$$T - T_w = \frac{2-\alpha}{\alpha}\,\frac{\mu}{2}\left(\frac{2\pi}{p\rho}\right)^{1/2}\frac{15}{8}\frac{\partial T}{\partial y} \qquad \text{at } y = 0 \qquad\qquad (3.5)$$

Let us mention in passing that the factor $\mu/\sqrt{p\rho}$ is proportional to the length of the mean free path of the gas molecules.

Experimental research results confirm the correctness of (3.4) and (3.5) for motion at subsonic velocities in rarefied gases.

To secure the boundary conditions for motion in a rarefied gas at $M > 1$, we make use of the fact that the derivation of the corresponding equations of motion and energy for a rarefied gas are related to the distribution function in its third approximation; developing Maxwell's method mentioned above, we substitute f_3 in Eq. (3.1) and (3.2).

In view of the fact that the function f_3 is not cited in explicit form in any other paper, we shall take the liberty of writing it in full here:

$$f = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\frac{-mv^2}{2kT}\left\{1 - \frac{3\mu}{2pT}\left(\frac{mv^2}{2kT} - \frac{5}{2}\right)\left(v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) - \right.$$

$$-\frac{\mu m}{2pkT}\left\{v_xv_y\left[2\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - Q_{12}\right] + v_xv_z\left[2\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - Q_{13}\right] + \right.$$

$$+ v_yv_z\left[2\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) - Q_{23}\right] + v_x^2\left[\frac{2}{3}\left(2\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} - \frac{\partial u_z}{\partial z}\right) - Q_{11}\right] +$$

$$+ v_y^2\left[\frac{2}{3}\left(2\frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) - Q_{22}\right] +$$

$$\left.\left. + v_z^2\left[\frac{2}{3}\left(2\frac{\partial u_z}{\partial z} - \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) - Q_{33}\right]\right\}\right\} \tag{3.6}$$

Here $\underline{n}$ is the number of molecules per unit volume while Q_{ik} are functions of the second derivatives and of their products with the first derivatives, as follows:

$$Q_{12} = k_1 2\frac{d}{dt}\frac{\mu}{p}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - k_1\frac{2\mu}{p}\left(\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y} - \right.$$

$$\left. - \frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial z}\frac{\partial u_y}{\partial z}\right) + k_2\frac{3\mu}{\rho pT}\left(\frac{\partial T}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial T}{\partial y}\frac{\partial p}{\partial x}\right) +$$

$$+ k_3\frac{3}{pT}\left(\frac{\partial}{\partial y}\mu\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\mu\frac{\partial T}{\partial y}\right) - k_4\frac{6\mu}{pT^2}\frac{\partial T}{\partial x}\frac{\partial T}{\partial y} + k_5\frac{2\mu}{3p}\left[3\left(\frac{\partial u_x}{\partial z} + \right.\right.$$

$$\left.\left. + \frac{\partial u_z}{\partial x}\right)\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) - 2\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\left(2\frac{\partial u_z}{\partial z} - \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right)\right] \tag{3.7}$$

Q_{13} and Q_{23} are derived from Q_{12} by substituting $\underline{z}$ for $\underline{x}$, and so forth.

$$Q_{11} = k_1\frac{2}{3}\frac{d}{dt}\frac{\mu}{p}\left(2\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} - \frac{\partial u_z}{\partial z}\right) - k_1\frac{\mu}{p}\left[\left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2 - \right.$$

$$\left. - \left(\frac{\partial u_x}{\partial y}\right) - \left(\frac{\partial u_x}{\partial z}\right)^2\right] + k_2\frac{\mu}{\rho pT}\left(2\frac{\partial T}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial T}{\partial y}\frac{\partial p}{\partial y} - \frac{\partial p}{\partial z}\frac{\partial T}{\partial z}\right) +$$

$$+ k_3\frac{1}{pT}\left(2\frac{\partial}{\partial x}\mu\frac{\partial T}{\partial x} - \frac{\partial}{\partial y}\mu\frac{\partial T}{\partial y} - \frac{\partial}{\partial z}\mu\frac{\partial T}{\partial z}\right) - k_4\frac{\mu}{pT^2}\left[2\left(\frac{\partial T}{\partial x}\right)^2 - \right.$$

$$\left. - \left(\frac{\partial T}{\partial y}\right)^2 - \left(\frac{\partial T}{\partial z}\right)^2\right] + k_5\frac{\mu}{9p}\left[2\left(2\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} - \frac{\partial u_z}{\partial z}\right)^2 + 3\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)^2 + \right.$$

$$\left. + 3\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)^2 - 6\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial y}\right)^2 - 6\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_z}{\partial z}\right)^2\right] \tag{3.8}$$

Q$_{22}$ and Q$_{33}$ are readily secured from (3.8) substituting y for x and so forth.

Here k$_i$ is a constant that differs for different models of molecular interaction; for the Maxwellian molecules:

$$k_1 = k_3 = k_5 = 1, \qquad k_2 = k_4 = 0 \tag{3.9}$$

while for molecules that are solid elastic spheres:

$$K_1 = 1.014, \ K_2 = 0.227, \ K_3 = 0.806, \ K_4 = 0.330, \ K_5 = 0.842 \tag{3.10}$$

After calculating several integrals and making the appropriate transformations, we get:

$$u_x = \frac{2-\alpha}{\alpha} \ \frac{\mu}{2}\left(\frac{2\pi}{p\rho}\right)^{\frac{1}{2}} \frac{\partial u_x}{\partial y} \frac{1}{A} \qquad \text{at } y = 0 \tag{3.11}$$

$$T\frac{3A-1}{2A} = T_w + \frac{2-\alpha}{\alpha} \ \frac{\mu}{2}\left(\frac{2\pi}{p\rho}\right)^{\frac{1}{2}} \frac{15}{8}\frac{\partial T}{\partial y} \frac{1}{A} \qquad \text{at } y = 0 \tag{3.12}$$

Here

$$A = 1 + \left(\frac{1}{3}k_5 - k_1\right)\frac{\mu^2}{p^2}\left(\frac{\partial u_x}{\partial y}\right)^2 + k_3\frac{\mu}{p\rho T}\ \frac{1}{cy}\ \mu\ \frac{\partial T}{\partial y} - k_4\frac{\mu^2}{p\rho T^2}\left(\frac{\partial T}{\partial y}\right)^2 \tag{3.13}$$

It is evident that when A = 1 Eq. (3.12) and (3.13) are transformed into Eq. (3.4) and (3.5), respectively.

It is of some interest to ascertain the extent to which A differs from unity. To do this we make use of the expression for the normal pressure gradient (1.11), writing it for Maxwellian molecules; then, from (1.11), (1.15), and (1.5), we get:

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}2\mu^2\left\{\frac{1}{3p}\left(\frac{\partial u_x}{\partial y}\right)^2 - \frac{1}{\rho T}\ \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho T^2}\left(\frac{\partial T}{\partial y}\right)^2\right\} \tag{3.14}$$

Integrating (3.14) and making use of the equation of state $p = \rho RT$, we get:

$$p - p_0 = \frac{2\mu^2}{p}\left[\frac{1}{3}\left(\frac{\partial u_x}{\partial y}\right)^2 - R\frac{\partial^2 T}{\partial y^2} - \frac{R}{T}\left(\frac{\partial T}{\partial y}\right)^2\right] \tag{3.15}$$

Now we write the expression for A for Maxwellian molecules; substituting (3.9) in (3.13) and making use of the equation of state, we get:

$$A = 1 - \frac{\mu^2}{p^2}\left[\frac{1}{3}\left(\frac{\partial u_x}{\partial y}\right)^2 - R\frac{\partial^2 T}{\partial y^2} - \frac{R}{T}\left(\frac{\partial T}{\partial y}\right)^2\right] \tag{3.16}$$

Comparing (3.15) and (3.16), we find that:

$$A = 0.5(1 + p_0/p)$$

<div style="text-align:right">(3.17)</div>

i.e. $A = 1$ when $p = p_0$, and $A = 0.5$ when $p \gg p_0$. Hence, if Eq. (1.11) is correct, the value of $\underline{A}$ is close to unity, varying over a comparatively narrow range. It may be expected that $\underline{A}$ will differ but little from unity for real gases as well, as the model of Maxwellian molecules satisfactorily describes their behavior.

Solving the equations of motion and energy for the boundary conditions (3.11) and (3.12) is an extremely complicated matter, all the more so as we first must know the distribution of velocity and temperature close to the surface of the body. The following method of successive approximations may be indicated: We solve Eq. (1.10) - (1.17) for the boundary conditions (3.4) and (3.5) and obtain the velocity and temperature profiles; and we again solve Eq. (1.10) - (1.17), but with the boundary conditions (3.11) and (3.12). Obtaining more accurate values of velocity and temperature, we substitute them in $\underline{A}$ and so forth. This is an extremely laborious job, however, and it is hardly justifiable; it is much more sensible to make use of simpler boundary conditions, assuming that $A = 1$.

In conclusion, I wish to thank A. I. Gubanov for setting up this problem and for his valuable comments during its solution.

LITERATURE CITED

1. Maxwell, J. C., "On Stresses in Rarefied Gases Arising from Inequalities of Temperature," The Scientific Papers, Vol. 11, 681 (1890).

2. Smoluchowski, P. "Further Studies on the Temperature Discontinuity in Gases," Wiener Berichte, 108, Bid. 11a (1899).

3. Chapman, S., and T. G. Cowling. The Mathematical Theory of Non-Uniform Gases. Cambridge, 1939.

4. Tsien, H. "The superaerodynamics," J. Aeronautical Sciences (Russian translation: Gazovaya Dinamika, a collection of articles, p. 310. Moscow: Foreign Literature Publishing House, 1950).

5. Popov, S. G., "Similitude in the aeromechanics of rarefied gases," Vestnik Moskovskogo universiteta, No. 8, p. 13 (1949).

6. Popov, S.G., "On the integral of one-dimensional flow in its third approximation," Vestnik Moskovskogo Universiteta, No. 2, p. 7 (1950).

7. Krzywoblocki, M. Z.,"On the two-dimensional laminar boundary layer equations for hypersonic flow in continuum and in rarefied gases," J. Aeronaut. Soc. India, 5, No. 1: 1-13, (1953).

8. Wang-Chang, C. S., and G. E. Uhlenbeck, "On Transport phenomena in rarefied gases," Dept. Eng. Res., Univ. Michigan, App. Phys. Labor., Rep. NO APL/JHU CM-443, UMN-3, February, 1948.

9. Krzywoblocki, M. Z., "The possible particular solutions of the laminar boundary layer on flat plate for hypersonic flow in continuum and in rarefied gases," J. Aeronaut. Soc. India 5, No. 2: 23-35 (1953).

10. Goody, R. M. Physics of the Stratosphere. Cambridge, 1954.

11. V. M., " Pressure, density, and temperature of the earth's atmosphere at heights up to 160 km," Uspekhi fizich nauk 48, No. 4: 609 (1952).

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